

Application: $\epsilon(\omega) - 1$

Recall that $\epsilon(\omega) \rightarrow 1$ as $\omega \rightarrow \infty$ and $\epsilon(\omega)$ is analytic in the upper complex ω -plane. For dielectrics, also analytic on the real ω -axis.

Hence

$$\operatorname{Re} \epsilon(\omega) = 1 + \frac{1}{\pi} P \int_{-\infty}^{\infty} \frac{\operatorname{Im} \epsilon(\omega')}{\omega' - \omega} d\omega'$$

$$\operatorname{Im} \epsilon(\omega) = -\frac{1}{\pi} P \int_{-\infty}^{\infty} \frac{\operatorname{Re} \epsilon(\omega') - 1}{\omega' - \omega} d\omega'$$

Recall that $\epsilon(-\omega) = \epsilon^*(\omega^*)$.

That is, if $\omega \in \mathbb{R}$, $\epsilon(-\omega) = \epsilon^*(\omega)$

$$\operatorname{Re} \epsilon(-\omega) = \operatorname{Re} \epsilon(\omega) \quad \text{even function}$$

$$\operatorname{Im} \epsilon(-\omega) = -\operatorname{Im} \epsilon(\omega) \quad \text{odd function}$$

$$\operatorname{Re} \epsilon(\omega) = 1 + \frac{2}{\pi} P \int_0^\infty \frac{\omega' \operatorname{Im} \epsilon(\omega')}{\omega'^2 - \omega^2} d\omega'$$

$$\operatorname{Im} \epsilon(\omega) = -\frac{2\omega}{\pi} P \int_0^\infty \frac{\operatorname{Re} \epsilon(\omega') - 1}{\omega'^2 - \omega^2} d\omega'$$

Kramers-Kronig relations

relates dispersive of the medium to the absorptive properties.

Remark: to apply to conductor ($\sigma \gg 1$),
then consider

$$\epsilon(\omega) \longrightarrow \epsilon(\omega) - \frac{4\pi i\sigma}{\omega}$$

Check out Jackson for "sum rules".

Phase velocity and group velocity

Recall the dispersion relation

$$\vec{k}^2 = \epsilon \mu \frac{\omega^2}{c^2} = n^2 \frac{\omega^2}{c^2}$$

$$n = n(\omega)$$

$$k(\omega) = \frac{\omega}{c} n(\omega)$$

$$\vec{E} = \vec{E}_0 e^{i(\vec{k} \cdot \vec{x} - \omega t)}$$

$$\vec{k} \cdot \vec{x} - \omega t = \vec{k} \cdot [\vec{x}_0 + \frac{\omega}{k} \hat{k} t] = \vec{k} \cdot \vec{x}_0 = \text{constant}$$

$$\vec{x} = \vec{x}_0 + \frac{\omega}{k} \hat{k} t = \vec{x}_0 + \vec{v}_p t$$

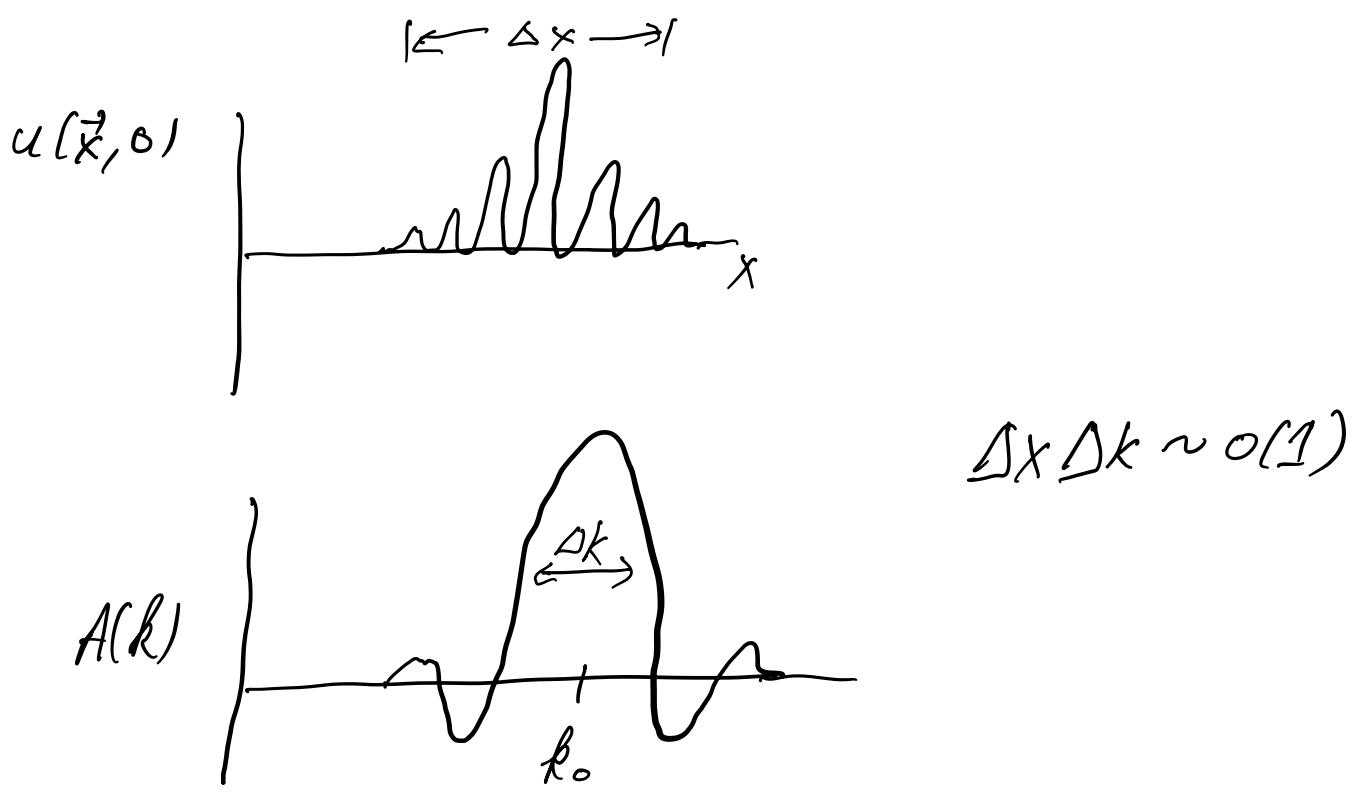
$$\text{where } \vec{v}_p = \frac{\omega}{k} \hat{k} = \frac{ck}{n(\omega)}$$

In the presence of absorption

$$e^{i(\vec{k} \cdot \vec{x} - \omega t)} = e^{i(\vec{k} \cdot \vec{x} - \omega t)} e^{-\vec{k}_I \cdot \vec{x}}$$

In practical applications, a wave includes multiple frequencies.

$$u(\vec{x}, t) = \frac{1}{(2\pi)^3 v_2} \int d^3 k A(\vec{k}) e^{i[\vec{k} \cdot \vec{x} - \omega(\vec{k}) t]}$$



$$\omega(\vec{k}) = \omega(\vec{k}_0) + \left(\frac{\partial \omega}{\partial k_i} \right)_{\vec{k}=\vec{k}_0} (\vec{k}-\vec{k}_0)_i + \dots$$

Group velocity

$$\vec{v}_g = \vec{\nabla}_{\vec{k}} \omega(\vec{k}) \Big|_{\vec{k}=\vec{k}_0}$$

$$\omega(\vec{k}) = \omega_0 + \vec{v}_g \cdot (\vec{k} - \vec{k}_0)$$

Then,

$$u(\vec{x}, t) = \frac{1}{(2\pi)^{3/2}} e^{i(\vec{k}_0 \cdot \vec{v}_g - \omega_0)t} \int d^3k A(\vec{k}) \times e^{i\vec{k} \cdot (\vec{x} - \vec{v}_g t)}$$

$$u(\vec{x}, t) = u(\vec{x} - \vec{v}_g t, 0) e^{i(\vec{k}_0 \cdot \vec{v}_g - \omega_0)t}$$

Modulo the phase factor, the pulse travels along undisturbed in shape with group velocity \vec{v}_g .

$$V_g^{-1} = \frac{dk}{d\omega} = \frac{1}{c} \left[n(\omega) + \omega \frac{dn}{d\omega} \right]$$

after
using
 $k = \frac{\omega}{c} n(\omega)$

$$V_g = \frac{c}{n(\omega) + \omega \frac{dn}{d\omega}}$$

If n is independent of ω , then $V_g = V_p = \frac{c}{n}$.

For dispersive medium, $V_g \neq V_p$

Normal dispersion

$$n > 1, \frac{dn}{d\omega} > 0 \Rightarrow V_g < V_p < c$$

Anomalous dispersion

$$\frac{dn}{d\omega} < 0 \Rightarrow V_g > V_p$$

Cases exist where $V_p > c$ and even $V_g > c$!

Relativity and Electromagnetism

Space + time \rightarrow spacetime

$\vec{x}, t \rightarrow$ four-vector x^μ

Contravariant vector

$$x^\mu = (x^0; x^1, x^2, x^3) = (x^0; \vec{x})$$

where $x^0 = ct$

Metric tensor

$$g_{\mu\nu} = g^{\mu\nu} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix}$$

"mostly minus"

Convention

Greek letters $\mu, \nu = 0, 1, 2, 3$

Latin letters $i, j = 1, 2, 3$

Some books use
 $g_{\mu\nu} = \text{diag}(-1, 1, 1, 1)$
"mostly plus"

$$g^{\mu\nu} = (g_{\mu\nu})^{-1}$$

Einstein summation convention

$$\sum_\nu g_{\mu\nu} g^{\nu\lambda} = \delta_\mu^\lambda = \begin{cases} 1 & \mu = \lambda \\ 0 & \mu \neq \lambda \end{cases}$$

↑ Kronecker delta

can drop \sum_ν and simply write

$$g_{\mu\nu} g^{\nu\lambda} = \delta_\mu^\lambda$$

two types of indices:

dummy indices: ✓

free indices: μ, λ (must match on both sides
of the equation)

$$\left. \begin{aligned} & (g_{\mu\nu} g^{\nu\lambda} = \delta_\mu^\lambda) \\ & \text{i.e. } g_\mu^\lambda = \delta_\mu^\lambda \end{aligned} \right\}$$

In relativity, index pairs that are implicitly summed over must involve one lower index and one upper index.

The length of a four-vector

$$\begin{aligned} x^2 &= g_{\mu\nu} x^\mu x^\nu = (x^0)^2 - |\vec{x}|^2 \\ &= c^2 t^2 - |\vec{x}|^2 \end{aligned}$$

dot product of a^μ, b^ν

$$g_{\mu\nu} a^\mu b^\nu$$

Introduce the covariant vector

$$x_\mu = g_{\mu\nu} x^\nu = (ct; -\vec{x})$$

$$\begin{aligned} \text{i.e. } x_0 &= x^0 \\ x_i &= -x^i \end{aligned}$$

Use $g_{\mu\nu}$ to lower the index ν

$$\text{Likewise } x^\mu = g^{\mu\nu} x_\nu$$

Lorentz transformations

change reference frames

$$x'^\mu = \Lambda^\mu{}_\nu x^\nu$$

↑
 a matrix μ rows
 ν columns

$$x'^\mu x'_\mu = x^\nu x_\nu$$

$$g_{\mu\lambda} x'^\mu x'^\lambda = g_{\nu\sigma} x^\nu x^\sigma$$

$$(\Lambda^\mu{}_\nu g_{\mu\lambda} \Lambda^\lambda{}_\sigma - g_{\nu\sigma}) x^\nu x^\sigma = 0$$

must be true for arbitrary x^ν

$$\boxed{\Lambda^\mu{}_\nu g_{\mu\lambda} \Lambda^\lambda{}_\sigma = g_{\nu\sigma}}$$

defining equation for a Lorentz transformation

or equivalently

$$\boxed{\Lambda^T G \Lambda = G}$$

$$G = \text{diag}(1, -1, -1, -1)$$

recall how matrices multiply

$$(\Lambda^T)_\nu{}^\mu = \Lambda^\mu{}_\nu$$

$$(AB)_{ij} = \sum_k A_{ik} B_{kj}$$

(If G were the identity matrix, then $\Lambda^T G \Lambda = G \Rightarrow \Lambda^T \Lambda = I$)

Λ is a pseudo-orthogonal matrix

$O(1, 3)$ mostly minus

$O(3, 1)$ mostly plus $G \rightarrow -G$

$$O(1, 3) \cong O(3, 1)$$

Take determinant of $\Lambda^T G \Lambda = G$

$$\det G = -1$$

$$\det \Lambda^T = \det \Lambda$$

$$\Rightarrow (\det \Lambda)^2 = 1$$

$$\boxed{\det \Lambda = \pm 1}$$

Return to $\Lambda^\mu_{\nu} g_{\mu\nu} \Lambda^\lambda_{\sigma} = g_{\nu\sigma}$

Put $\nu = \sigma = 0$ (free indices). Then $g_{00} = 1$ and

$$(\Lambda^0_0)^2 - (\Lambda^1_0)^2 - (\Lambda^2_0)^2 - (\Lambda^3_0)^2 = 1$$

$$(\Lambda^0_0)^2 = 1 + (\Lambda^1_0)^2 + (\Lambda^2_0)^2 + (\Lambda^3_0)^2$$

$$\Rightarrow (\Lambda^0_0)^2 \geq 1$$

$$\text{That is } \boxed{\Lambda^0_0 \geq 1 \text{ or } \Lambda^0_0 \leq -1}$$

There are four classes of Lorentz transformations:

$\det \Lambda = 1, \Lambda^0_0 \geq 1 \leftarrow$ proper or orthochronous

$\det \Lambda = 1, \Lambda^0_0 \leq -1$

$\det \Lambda = -1, \Lambda^0_0 \geq 1$

$\det \Lambda = -1, \Lambda^0_0 \leq -1 \right\}$ improper